

On a property of the n -dimensional cube

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We show that in any subset of the vertices of n -dimensional cube that contains at least $2^{n-1} + 1$ vertices ($n \geq 4$), there are four vertices that induce a claw, or there are eight vertices that induce the cycle of length eight.

1. Introduction and definitions

We consider finite graphs $G = (V, E)$ with vertex set V and edge set E . The graphs contain no multiple edges or loops. The n -dimensional cube will be denoted by Q_n , and claw is the complete bipartite graph $K_{1,3}$. Moreover, the vertex of degree three in the claw is called a claw-center. Non-defined terms and concepts can be found in [1].

The main result of the paper is the following:

Theorem 1 *Let $n \geq 4$ and let $V' \subseteq V(Q_n)$. If $|V'| \geq 2^{n-1} + 1$, then at least one of the following two conditions holds:*

- (a) *there are four vertices in V' that induce a claw;*
- (b) *there are eight vertices that induce a simple cycle.*

Proof. Our proof is by induction on n . Suppose that $n = 4$. Clearly, without loss of generality, we can assume that $|V'| = 9$. Consider the following partition of the vertices of Q_4 :

$$V_1 = \{(0, \alpha_2, \alpha_3, \alpha_4) : \alpha_i \in \{0, 1\}, 2 \leq i \leq 4\}, V_2 = \{(1, \alpha_2, \alpha_3, \alpha_4) : \alpha_i \in \{0, 1\}, 2 \leq i \leq 4\}.$$

Clearly, the subgraphs of Q_4 induced by V_1 and V_2 are isomorphic to Q_3 . Define:

$$V'_1 = V_1 \cap V', V'_2 = V_2 \cap V'.$$

We will assume that $|V'_1| \geq |V'_2|$. We will complete the proof of the base of induction, by considering the following cases:

Case 1: $|V'_1| = 8$ and $|V'_2| = 1$. Clearly, any vertex from V'_1 is a claw-center.

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Case 2: $|V'_1| = 7$ and $|V'_2| = 2$. It is not hard to see that V'_1 contains a claw-center.

Case 3: $|V'_1| = 6$ and $|V'_2| = 3$. Again, it is a matter of direct verification that V'_1 contains a claw-center.

Case 4: $|V'_1| = 5$ and $|V'_2| = 4$. Consider the subgraph G_1 of Q_4 induced by V'_1 . Clearly, if G_1 contains a vertex of degree three, then this vertex is a claw-center. Thus, any vertex in G_1 has degree at most two. It is not hard to see that this implies that G_1 contains no isolated vertex. Moreover, since $|V'_1| = 5$, we have that G_1 is a connected graph, and therefore it is the path of length four.

Now, let a_1, a_2, a_3 be the internal vertices of G_1 , and let b_1, b_2 be the end-vertices of G_1 . Clearly, we can assume that neither of a_1, a_2, a_3 has a neighbour in V'_2 . Since $|V_2| = 8$ and $|V'_2| = 4$, we have that there are five possibilities for V'_2 . We invite the reader to check that in four of these cases one can find a claw-center in V'_2 , and in the final case V' has a vertex z such that $V' \setminus \{z\}$ induces a simple cycle.

Now, let us assume that the statement is true for $n - 1$, and let $V' \subseteq V(Q_n)$ be a subset with $|V'| \geq 2^{n-1} + 1$. Consider the following partition of the vertices of Q_n :

$$V_1 = \{(0, \alpha_2, \dots, \alpha_n) : \alpha_i \in \{0, 1\}, 2 \leq i \leq n\}, V_2 = \{(1, \alpha_2, \dots, \alpha_n) : \alpha_i \in \{0, 1\}, 2 \leq i \leq n\}.$$

Clearly, the subgraphs of Q_n induced by V_1 and V_2 are isomorphic to Q_{n-1} . Moreover, it is not hard to see that at least one of the following two inequalities is true: $|V_1 \cap V'| \geq 2^{n-2} + 1$ and $|V_2 \cap V'| \geq 2^{n-2} + 1$. Thus the proof follows from the induction hypothesis. \square

For the case of $n = 3$ we have:

Proposition 1 *Let $V' \subseteq V(Q_3)$ and let $|V'| \geq 6$. Then at least one of the following two conditions holds:*

- *there are four vertices in V' that induce a claw;*
- *there are six vertices in V' that induce a simple cycle.*

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REFERENCES

1. F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.